Okanagan College

Math 112 (071) Fall 2009

Term Test Three – Problems & Solutions

Instructor: Clint Lee

Wednesday, November 26

Student Name: ____________________ Total Marks: ________ / 40

Instructions. Do all parts of all 8 questions. Show all work and give explanations where required. You may receive part marks for a question if your work is correct even if the final answer is incorrect. However, if your answer is incorrect and no work or explanation is given, you will receive no marks. The number of points for each question is given in the left margin, total 40.

In problems 1 through 3 give a brief answer. You will be marked only on your answer, not on your work.

1. Differentiate each function. Do not simplify

[2] (a) $f(x) = \sin\left(x^2\right)$

Solution

$$f'(x) = \cos\left(x^2\right)(2x)$$

[2] (b) $s = \ln \sqrt[3]{x^3+1}$

Solution

First write s as

$$s = \ln\left(x^3+1\right)^{1/3} = \tfrac{1}{3}\ln\left(x^3+1\right)$$

Then

$$\frac{ds}{dx} = \frac{1}{3}\left(\frac{1}{x^3+1}\right)\left(3x^2\right)$$

[2] 2. Suppose that x and y satisfy the equation $xe^y = y^2 + 1$ and that y is decreasing at the rate of 2 units per minute. Find the time rate of change of x when $x = 1$ and $y = 0$. Give your answer in exact form (no decimals).

Solution

This is a related rates problem with:
Given:

$$\frac{dx}{dt} = -2\,\text{units/min}$$

Find:

$$\frac{dy}{dt} \text{ when } x = 1 \text{ and } y = 0$$

The relation is given. So take derivative with respect to t treating both x and y as functions of t. This gives

$$\left(\frac{dx}{dt}\right)e^y + xe^y\left(\frac{dy}{dt}\right) = 2y\left(\frac{dy}{dt}\right)$$

Solving for the required rate $\frac{dy}{dt}$ gives

$$(2y - xe^y)\left(\frac{dy}{dt}\right) = e^y\left(\frac{dx}{dt}\right) \Rightarrow \frac{dy}{dt} = \frac{e^y}{2y - xe^y}\left(\frac{dx}{dt}\right)$$

Plugging in given rate and information at the instant of interest gives

$$\left.\frac{dy}{dt}\right|_{x=1,y=0} = \frac{e^0}{2\,(0) - (1)\,e^0}\,(-2) = 2\,\text{units/min}$$

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[2] 3. Verify that the linear approximation to $f(x) = \ln(x-2)$ near $x = 3$ is

$$\ln(x-2) \approx x - 3$$

Solution

The linear approximation at $x = 3$ is

$$f(x) \approx f(3) + f'(3)(x-3)$$

Here $f(3) = \ln(3-2) = \ln(1) = 0$ and

$$f'(x) = \frac{1}{x-2} \Rightarrow f'(3) = \frac{1}{3-2} = 1$$

Hence

$$\ln(x-2) \approx 0 + (1)(x-3) = x - 3$$

4. Differentiate each function. Simplify only if required.

[3] (a) $f(x) = x^2 \sin^3 2x$

Solution

Write the function as

$$f(x) = x^2 [\sin(2x)]^3$$

Then using the Product Rule and Chain Rule gives

$$f'(x) = 2x [\sin(2x)]^3 + x^2 \left\{3 [\sin(2x)]^2 [\cos(2x)(2)]\right\}$$

Simplifying (not required here) gives

$$f'(x) = 2x \sin^3 2x^3 + 6x^2 \sin^2 2x \cos 2x = 2x \sin^2 2x (\sin 2x + 3x \cos 2x)$$

[4] (b) $y = \arctan\left(\sqrt{x^2 - 1}\right)$. Simplify.

Solution

Using the Chain Rule twice gives

$$\frac{dy}{dx} = \frac{1}{1 + \left(\sqrt{x^2-1}\right)^2} \left[\left(\frac{1}{2}\right)\left(x^2 - 1^{-1/2}\right)(2x)\right]$$

Simplifying gives

$$\frac{dy}{dx} = \left(\frac{1}{1 + x^2 - 1}\right)\left(\frac{x}{\sqrt{x^2-1}}\right) = \left(\frac{1}{x^2}\right)\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{x\sqrt{x^2-1}}$$

[3] (c) $g(t) = \dfrac{\sec t}{\tan t + 1}$

Solution

Using the Quotient Rule gives

$$g'(t) = \frac{\sec t \tan t (\tan t + 1) - \sec t \left(\sec^2 t\right)}{(\tan t + 1)^2}$$

Simplifying gives (required on assignment)

$$g'(t) = \frac{\sec t \left(\tan^2 t + \tan t - \sec^2 t\right)}{(\tan t + 1)^2} = \frac{\sec t (\tan t - 1)}{(\tan t + 1)^2}$$

[4] 5. (a) Consider the equation

$$2x\sin y - y\cos x = x + y$$

Find $\dfrac{dy}{dx}$ for any function defined by this equation.

Solution

Differentiating both sides of the equation with respect to x treating y as a function of x gives

$$2\sin y + (2x\cos y)\,y' - \left[y'\cos x + y\,(-\sin x)\right] = 1 + y'$$

Solving for y' gives

$$(2x\cos y - \cos x - 1)\,y' = 1 - 2\sin y - y\sin x \Rightarrow \frac{dy}{dx} = y' = \frac{1 - 2\sin y - y\sin x}{2x\cos y - \cos x - 1}$$

[3] (b) Verify that the point $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ lies on the graph of the equation above. Find the equation of the line tangent to the graph of the equation above at the point $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$.

Solution

To verify that the point $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ lies on the graph of the equation plug $x = \frac{1}{2}\pi$ and $y = \frac{1}{2}\pi$ into each side of the equation: to give

$$\text{LHS} = 2\left(\tfrac{1}{2}\pi\right)\sin\left(\tfrac{1}{2}\pi\right) - \left(\tfrac{1}{2}\pi\right)\cos\left(\tfrac{1}{2}\pi\right) = 2\left(\tfrac{1}{2}\pi\right)(1) - \left(\tfrac{1}{2}\pi\right)(0) = \pi$$

$$\text{RHS} = \tfrac{1}{2}\pi + \tfrac{1}{2}\pi = \pi$$

The two sides of the equation are equal at $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$. This verifies that the point lies on the graph. At the point $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{x=\frac{1}{2}\pi, y=\frac{1}{2}\pi} = \frac{1 - 2\sin\left(\frac{1}{2}\pi\right) - \left(\frac{1}{2}\pi\right)\sin\left(\frac{1}{2}\pi\right)}{2\left(\frac{1}{2}\pi\right)\cos\left(\frac{1}{2}\pi\right) - \cos\left(\frac{1}{2}\pi\right) - 1} = \frac{1 - 2 - \frac{1}{2}\pi}{2\,(0) - 0 - 1} = 1 + \tfrac{1}{2}\pi$$

Hence, the equation of the tangent line is

$$y - \tfrac{1}{2}\pi = \left(1 + \tfrac{1}{2}\pi\right)\left(x - \tfrac{1}{2}\pi\right) \Rightarrow y = \left(1 + \tfrac{1}{2}\pi\right)\left(x - \tfrac{1}{2}\pi\right) + \tfrac{1}{2}\pi = \left(1 + \tfrac{1}{2}\pi\right)x - \tfrac{1}{4}\pi^2$$

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[4] 6. Use logarithmic differentiation to find $\dfrac{dy}{dx}$ for $y = (x^2+1)^{\sin x}$.

Solution

Take logarithms of both sides to give

$$\ln y = \sin x \ln\left(x^2 + 1\right)$$

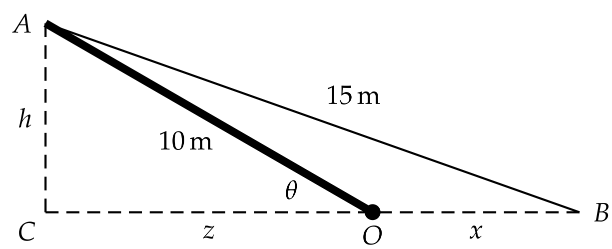
Take derivative of both sides to give

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln\left(x^2+1\right) + \sin x\left(\frac{1}{x^2+1}\right)(2x) = \cos x \ln\left(x^2+1\right) + \frac{2x\sin x}{x^2+1}$$

Solve for $\dfrac{dy}{dx}$ and substitute $y = (x^2+1)^{\sin x}$ to give

$$\frac{dy}{dx} = \left(x^2+1\right)^{\sin x}\left[\cos x \ln\left(x^2+1\right) + \frac{2x\sin x}{x^2+1}\right]$$

7. A 10-metre totem pole will be raised by anchoring one end, labeled O in the diagram, and attaching a 15-metre rope to the other end, A. The rope is laid out along the totem pole and extended beyond the anchored end to point B. The end at B is then pulled along the ground at 20 cm/s directly away from the anchored end of the totem pole raising the end A off the ground.



[2] (a) Use Pythagoras theorem for each of the two right triangles in the diagram to show that

$$x^2 + 2xz = 125$$

Solution

Label the point on the ground directly below the moving end of the pole C, and apply Pythagoras' theorem to the triangles AOC and ABC to give

$$h^2 + z^2 = 10^2 = 100$$
$$h^2 + (z + x)^2 = h^2 + z^2 + 2zx + x^2 = 15^2 = 225$$

Using the first equation in the second gives

$$100 + 2zx + x^2 = 225 \Rightarrow x^2 + 2xz = 125$$

as required.

Problem 7 continued ...

...Problem 7 continued

[4] (b) Twenty five seconds after the rope starts moving the sun is directly overhead. At this time the distance z in the diagram above is the length of the shadow cast by the totem pole. Find the rate at which the length of the shadow cast by the totem pole is lengthening 25 seconds after the rope starts moving. Hint: Note that the dimension x is not zero when the rope stars moving.

Solution

This is a related rates problem for which:
Given:

$$\frac{dx}{dt} = 20\,\text{cm/sec} = 0.20\,\text{m/sec}$$

Find:

$$\frac{dz}{dt}\ \text{25 seconds after the rope starts moving}$$

The relation was determined to be

$$x^2 + 2xz = 125$$

Taking the derivative with respect to t treating both x and z as functions of t gives

$$2x\frac{dx}{dt} + 2\left[\left(\frac{dx}{dt}\right)z + x\left(\frac{dz}{dt}\right)\right] = 0 \Rightarrow \frac{dz}{dt} = -\left(\frac{x+z}{x}\right)\frac{dx}{dt}$$

Just as the rope starts moving the 15-metre rope extends 5 metres past the end of the pole. So at this time $x = 5$. Hence, twenty five seconds after the rope starting moving $x = 5 + 0.2(25) = 10\,\text{m}$. At this time

$$z = \frac{125 - x^2}{2x} = \frac{125 - 100}{20} = \frac{5}{4}$$

Plugging this value and the given value of x together with the given rate into the expression above gives

$$\frac{dz}{dt} = -\left(\frac{10 + \frac{5}{4}}{10}\right)(0.2) = -\frac{9}{40} = -0.225\,\text{m/s} = -22.5\,\text{cm/s}$$

This means that the length of the shadow is in fact getting less. So the shadow is shortening.

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8. In forestry the volume of wood in a tree is estimated by measuring the circumference of the tree at the base and computing the radius of the base of the tree knowing the circumference. Then the volume of the tree is computed by assuming that the tree is a cone whose height is approximately 100 times the radius of the base.

[2] (a) Given that the volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$, find a formula for the volume estimate for a tree whose circumference at the base is C. Express the volume as function of C.

Solution

If the height of the tree is 100 time the radius of the base, then $h = 100r$ and the volume of the tree is

$$V = \frac{1}{3}\pi r^2\,(100r) = \frac{100}{3}\pi r^3$$

The circumference of a circle of radius r is $C = 2\pi r$. So that

$$r = \frac{C}{2\pi}$$

and the volume in terms of the circumference is

$$V = \frac{100}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{100}{3}\pi\left(\frac{C^3}{8\pi^3}\right) = \frac{25}{6\pi^2}C^3$$

Problem 8 continued ...

. . . Problem 8 continued

[3] (b) The base circumference of a tree is measured to be 80 cm with a maximum error of 1 cm. Use differentials to estimate the maximum possible error in the calculated volume of the tree. Give both the absolute and relative errors.

Solution

The absolute error in the computed value of the volume, in cm^3, is dV when $C = 80$ and $dC = 1$. Here

$$dV = \frac{25}{6\pi^2}\left(3C^2\,dC\right) = \frac{25}{2\pi^2}C^2\,dC$$

So the absolute error is

$$dV = \frac{25}{2\pi^2}\left(80^2\right)(1) = \frac{80000}{\pi^2} = 8105\,\text{cm}^3$$

The relative error is

$$\frac{dV}{V} = \frac{\left(\dfrac{25}{2\pi^2}C^2\,dC\right)}{\left(\dfrac{25}{6\pi^2}C^3\right)} = 3\left(\frac{dC}{C}\right) = 3\left(\frac{1}{80}\right) = 0.0375 = 3.75\%$$