Example 1

Let $p(x) = \sqrt{5 - x^2}$.

(a) Find the linearization of p at $x = 2$.

Solution

Find $p'(x)$:

$$p'(x) = \left(\frac{1}{2}\right)\left(5 - x^2\right)^{-1/2}(2x) = -\frac{x}{\sqrt{5 - x^2}}$$

Then find $p(2)$ and $p'(2)$:

$$p(2) = 1 \quad \text{and} \quad p'(2) = -\frac{2}{1} = -2$$

Finally, the linearization at $x = 2$ is:

$$L(x) = p(2) + p'(2)(x - 2) = 1 - 2(x - 2)$$

(b) Use the linearization in part (a) above to approximate $p(1.9)$.

Solution

The linearization is an approximation to $p(x)$ near $x = 2$. Since $x = 1.9$ is close to $x = 2$, the approximation should be reasonable:

$$p(1.9) \approx L(1.9) = 1 - 2(1.9 - 2) = 1 - 2(-0.1) = 1.2$$

(c) Find the quadratic approximation

$$Q(x) = p(a) + p'(a)(x - a) + \tfrac{1}{2}p''(a)(x - a)^2$$

to the function $p(x)$ near $x = 2$ and use it to improve your estimate of $p(1.9)$.

Solution

Now find $p''(x)$.

$$\begin{aligned} p''(x) &= -\frac{\left(5 - x^2\right)^{1/2} - x\left(\frac{1}{2}\right)\left(5 - x^2\right)^{-1/2}(-2x)}{(5 - x^2)} \\ &= -\frac{\left(5 - x^2\right)^{1/2} + \dfrac{x^2}{\left(5 - x^2\right)^{1/2}}}{(5 - x^2)} \\ &= -\frac{5 - x^2 + x^2}{\left(5 - x^2\right)^{3/2}} = -\frac{5}{\left(5 - x^2\right)^{3/2}} \end{aligned}$$

Then $p''(2) = -\dfrac{5}{1} = -5$ and the quadratic approximation is

$$Q(x) = p(2) + p'(2)(x - 2) + \tfrac{1}{2}p''(2)(x - 2)^2 = 1 - 2(x - 2) - \tfrac{5}{2}(x - 2)^2$$

Example 2

Use differentials (tangent line approximation) to estimate

(a) $\sqrt[4]{15.9}$ (b) $\sin 60.2^\circ$

Solution

For $\sqrt[4]{15.9}$ let $f(x) = \sqrt[4]{x} = x^{1/4}$ and use $x = 16$. Take $dx = -0.1$ to estimate the value at $x = 15.9$. Here

$$dy = \tfrac{1}{4}x^{-3/4}\,dx = \frac{dx}{4x^{3/4}}$$

Then for $x = 16$ and $dx = -0.1$ we have

$$dy = \frac{-0.1}{4(16)^{3/4}} = -\frac{0.1}{32} = -0.003125$$

So that

$$\sqrt[4]{15.9} \approx \sqrt[4]{16} + dy = 2 - 0.003125 = 1.996875 \quad \text{(exact value is 1.99686765)}$$

For $\sin 60.2^\circ$ let $f(x) = \sin x$ and take $x = 60^\circ = \frac{1}{3}\pi$. Take $dx = 0.2\left(\frac{\pi}{180}\right) = \frac{\pi}{900}$ to estimate the value at $x = 60.2^\circ$. Here

$$dy = \cos x\,dx$$

Then for $x = \frac{1}{3}\pi$ and $dx = \frac{\pi}{900}$ we have

$$dy = \sin\left(\tfrac{1}{3}\pi\right)\left(\frac{\pi}{900}\right) = \frac{\pi}{1800}$$

So that

$$\sin 60.2^\circ \approx \cos\left(\tfrac{1}{3}\pi\right) + dy = \tfrac{1}{2}\sqrt{3} + \tfrac{1}{1800}\pi = 0.867771 \quad \text{(exact value is 0.867765)}$$

Example 3

The frequency at which a stretched wire vibrates is given by the formula

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where f is the frequency of vibration in hertz, L is length of the wire in meters, T is the tension in newtons, and ρ is the linear mass density of the wire in kg/m. The string on a guitar has a length of 55 cm and is held at a tension of 25 N. The linear mass density of the string is 40 g/m.

(a) Consider the frequency f as a function of the tension T, with the length and density held constant, find the differential df.

Solution

Write the expression for f as

$$f = \frac{1}{2L\sqrt{\rho}}\left(\sqrt{T}\right) = \frac{1}{2L\sqrt{\rho}}(T)^{1/2}$$

Then

$$df = \frac{1}{2L\sqrt{\rho}}\left(\frac{1}{2}T^{-1/2}\,dT\right) = \frac{dT}{4L\sqrt{\rho T}}$$

Example 3 continued ...

...Example 3 continued

(b) Suppose that the tension in the wire is measured with a maximum possible error of 0.5 N. The other quantities are assumed to be known exactly. Use differentials to estimate the maximum possible error in the computed value of the frequency. Give both the absolute error and the relative error.

Solution

Plugging in the given values $L = 55\,\text{cm} = 0.55\,\text{m}$, $T = 25\,\text{N}$, and $\rho = 40\,\text{g/m} = 0.040\,\text{kg/m}$ together with $dT = 0.5\,\text{N}$. Then

$$df = \frac{0.5\,\text{N}}{4\,(0.55\,\text{m})\,\sqrt{(25\,\text{N})\,(0.040\,\text{kg/m})}} = 0.227273\,\text{s}^{-1}$$

To get the units use the fact the $1\,\text{N} = 1\,\text{kg}\,\text{m/s}^2$. This is the absolute error. To find the relative error we need the frequency. This is

$$f = \frac{1}{2\,(0.55\,\text{m})}\sqrt{\frac{25\,\text{N}}{0.040\,\text{kg/m}}} = 22.7273\,\text{s}^{-1}$$

Then the relative error is

$$\frac{df}{f} = \frac{0.227273}{22.7273} = 0.010 = 1\%$$

In general, the relative error is

$$\frac{df}{f} = \frac{\dfrac{dT}{4L\sqrt{\rho T}}}{\dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}}} = \frac{dT}{2T}$$

(c) Suppose that $y = \dfrac{u}{v}$ where both u and v are functions of x. Show that

$$dy = \frac{v\,du - u\,dv}{v^2}$$

Solution

Using the Quotient Rule gives

$$\frac{dy}{dx} = \frac{\left(\dfrac{du}{dx}\right)v - u\left(\dfrac{dv}{dx}\right)}{v^2} = \frac{\left(v\dfrac{du}{dx}\right) - u\left(\dfrac{dv}{dx}\right)}{v^2}$$

so that

$$dy = \frac{dy}{dx}\,dx = \frac{\left(v\dfrac{du}{dx}\right)dx - u\left(\dfrac{dv}{dx}\right)dx}{v^2} = \frac{v\,du - u\,dv}{v^2}$$

Example 3 continued ...

...Example 3 continued

(d) Suppose that the tension in the wire is measured with a maximum possible error of 0.5 N and the length is measured with a maximum possible error of 0.1 cm. The linear mass density is assumed to be known exactly. Use differentials, with the result from part (c) above, to estimate the maximum possible error in the computed value of the frequency. Give both the absolute error and the relative error.

Solution

Writing f as

$$f = \frac{1}{2\sqrt{\rho}} \frac{\sqrt{T}}{L}$$

so that

$$df = \frac{1}{2\sqrt{\rho}} \left(\frac{L\left(\frac{1}{2}T^{-1/2}\right)\,dT - T^{1/2}\,dL}{L^2} \right) = \frac{1}{4\sqrt{\rho T}} \left(\frac{L\,dT - T\,dL}{L^2} \right)$$

The maximum possible error in the computed value of the frequency is df when $T = 25\,\mathrm{N}$, $dT = 0.5\,\mathrm{N}$, $L = 55\,\mathrm{cm}$, and $dL = -0.001\,\mathrm{m}$. Note that we take dL to be negative to get the worst possible case for the error estimate. The absolute error is

$$df = \frac{1}{4\sqrt{(0.04)\,(25)}} \left(\frac{(0.55)\,(0.5) - (25)\,(-0.001)}{(0.55)^2} \right) = 0.248\,\mathrm{s}^{-1}$$

The relative error is

$$\frac{df}{f} = \frac{0.248}{22.7273} = 0.0109 = 1.1\%$$

Example 4

Martin borrows \$30000 to buy a new car at an interest rate of 0.4% per month with a term of 60 months. He wants to determine the effect of changes in interest rate on the total cost of borrowing money for the car. His accountant tells him that for an interest rate i per month with a term of n months, the total cost of borrowing money is

$$C = \frac{30000ni}{1-(1+i)^{-n}}$$

(a) Taking n to be a constant, calculate $\dfrac{dC}{di}$ for the formula above. You should not try to simplify this derivative too much.

Solution

Take out the $30000n$ as a constant, then apply the quotient rule. When taking the derivative of the denominator use the chain rule on the $(1+i)^{-n}$ term. This gives

$$\begin{aligned}\frac{dC}{di} &= 30000n\left(\frac{1-(1+i)^{-n} - i\left(0-(-n)(1+i)^{-n-1}\right)}{(1-(1+i)^{-n})^2}\right)\\ &= 30000n\left(\frac{1-(1+i)^{-n} - in(1+i)^{-n-1}}{(1-(1+i)^{-n})^2}\right)\end{aligned}$$

(b) Find the value of $\dfrac{dC}{di}$ for the interest rate and term of Martin's loan. Remember that 0.4% = 0.004. Use differentials, or a linear approximation, to estimate how much the total cost of the loan will change if the interest rate increases from 0.4% to 0.5%.

Solution

Plug $n = 60$ and $i = 0.4\% = 0.004$ into the expression in part (a) above:

$$\left.\frac{dC}{di}\right|_{i=0.004,n=60} = 30000(60)\left(\frac{1-(1.004)^{-60} - 0.004(60)(1.004)^{-61}}{(1-(1.004)^{-60})^2}\right) = 986627.90$$

This (huge) number gives the amount by which the total cost of the loan changes when the interest rate changes by one unit (1 = 100%) from 0.4%, that is, when the interest rate goes from 0.4% to 100.4%. To estimate the amount by which the total cost of the loan changes when the interest rate increase from 0.4% to 0.5% find dC when $i = 0.004$ (and $n = 60$) and $di = 0.001$. This gives

$$dC = \left.\frac{dC}{di}\right|_{i=0.004,n=60} di = 986627.90 \times 0.001 = \$986.63$$

(c) Taking i to be constant, calculate $\dfrac{dC}{dn}$ for the formula above.

Solution

Now take out $30000i$ and then apply the quotient rule. To take the derivative of the denominator use the general exponential rule

$$\frac{d}{dx}a^x = a^x \ln a$$

This gives

$$\begin{aligned}\frac{dC}{dn} &= 30000i\left(\frac{1-(1+i)^{-n} - n\left(-(1+i)^{-n}\ln(1+i)(-1)\right)}{(1-(1+i)^{-n})^2}\right)\\ &= 30000i\left(\frac{1-(1+i)^{-n} - n(1+i)^{-n}\ln(1+i)}{(1-(1+i)^{-n})^2}\right)\\ &= \frac{30000i\left[1-(1+i)^{-n}\left(1+n\ln(1+i)\right)\right]}{(1-(1+i)^{-n})^2}\end{aligned}$$

Example 4 continued ...

. . . Example 4 continued

(d) Find the value of $\frac{dC}{dn}$ for the interest rate and term of Martin's loan. Use differentials, or a linear approximation, to estimate how much the total cost of the loan will change when the term of the loan increases from 60 to 62 months.

Solution

Plug $n = 60$ and $i = 0.4\% = 0.004$ into the expression in part (c) above:

$$\left.\frac{dC}{di}\right|_{n=60,i=0.004} = \frac{30000(0.004)\left[1-(1.004)^{-60}\left(1+60\ln(1.004)\right)\right]}{\left(1-(1.004)^{-60}\right)^2} = 64.7812834$$

To estimate the amount by which the total cost of the loan changes when the term of the loan increases from 60 to 62 months find dC when $n = 60$ (and $i = 0.004$) and $dn = 2$. This gives

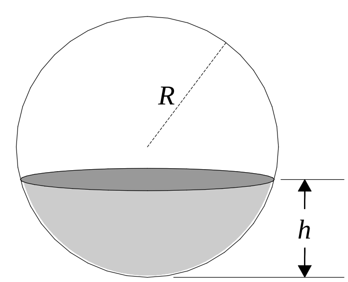
$$dC = \left.\frac{dC}{dn}\right|_{n=60,i=0.004} dn = 64.7812834 \times 2 = \$129.56$$

Example 5

A spherical segment is formed by cutting a sphere perpendicular to the axis. See the diagram. The volume of a spherical segment of depth h cut from a sphere of radius R is

$$V = \frac{\pi}{3}h^2\left(3R-h\right)$$

A spherical tank has a 40 cm radius. The depth of water in the tank is measured to 25 ± 0.1 cm. Estimate the error in the calculated volume of the water in the tank. Give both the absolute and relative errors.



Solution

The error in the calculated value of the volume is dV when $h = 25$ cm and $dh = 0.1$ cm. Write the volume as

$$V = \frac{\pi}{3}\left(3Rh^2 - h^3\right)$$

Then take the derivative with respect to h to give

$$\frac{dV}{dt} = \frac{\pi}{3}\left(6Rh - 3h^2\right) = \pi h\left(2R-h\right)$$

Plugging in gives the absolute error as

$$dV = \left.\frac{dV}{dh}\right|_{h=25\,\text{cm}} dh = \pi\,(25)\,[2\,(40)-25]\,(0.1) = 432\,\text{cm}^3$$

The computed volume when $h = 25$ cm is

$$V = \frac{\pi}{3}\,(25)^2\,[3\,(40)-25] = 81812\,\text{cm}^3$$

So the relative error is

$$\frac{dV}{V} = \frac{432}{81812} = 0.00528$$

So the relative error in the calculated value of the volume is about 0.53%.