Example 1

Differentiate and simplify $p(\theta) = \sin^4 3\theta$

Solution
Write the function as

$$p(\theta) = [\sin(3\theta)]^4$$

Then using the Chain Rule gives

$$p'(\theta) = 4[\sin(3\theta)]^3 [\cos(3\theta)](3) = 12\sin^3 3\theta \cos 3\theta$$

Example 2

Find $\dfrac{dq}{du}$ for $q = \dfrac{\tan^2 3u}{1+\sec 3u}$.

Solution
Using the Quotient Rule followed by the Chain Rule as in the previous problem gives

$$\begin{aligned}\frac{du}{dq} &= \frac{2\tan 3u \sec^2 3u(3)(1+\sec 3u) - \tan^2 3u \sec 3u \tan 3u(3)}{(1+\sec 3u)^2} \\ &= \frac{3\tan 3u \sec 3u \left(2\sec 3u + 2\sec^2 3u - \tan^2 3u\right)}{(1+\sec 3u)^2} \\ &= \frac{3\tan 3u \sec 3u \left(2\sec 3u + \sec^2 3u + 1\right)}{(1+\sec 3u)^2}\end{aligned}$$

since $\sec^2\theta = 1 + \tan^2\theta \Rightarrow \sec^2\theta - \tan^2\theta = 1$.

Example 3

Find the equation of the tangent line to the curve $y = x\tan x$ at the point on the graph where $x = \frac{\pi}{4}$. Give your answer in exact form, that is, without using decimal approximations. Recall that $\tan\left(\frac{\pi}{4}\right) = 1$ and $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

Solution
First find $\dfrac{dy}{dx}$. Using the Product Rule gives

$$\frac{dy}{dx} = \tan x + x\sec^2 x$$

Then the slope of the tangent line is

$$m = \left.\frac{dy}{dx}\right|_{x=\pi/4} = \tan\left(\tfrac{\pi}{4}\right) + \tfrac{\pi}{4}\sec^2\left(\tfrac{\pi}{4}\right) = 1 + \tfrac{\pi}{2}$$

since $\tan\left(\frac{\pi}{4}\right) = 1$ and $\sec\left(\frac{\pi}{4}\right) = \sqrt{2} \Rightarrow \sec^2\left(\frac{\pi}{4}\right) = 2$. Further, for $x = \frac{\pi}{4}$ the y-coordinate is $y = \frac{\pi}{4}$. So the equation of the tangent line is

$$y - \tfrac{\pi}{4} = \left(1 + \tfrac{\pi}{2}\right)\left(x - \tfrac{\pi}{4}\right) \Rightarrow y = \left(1 + \tfrac{\pi}{2}\right)x - \tfrac{\pi^2}{8}$$

Example 4

Let $y = e^t \cos t$.

(a) Find $\dfrac{dy}{dt}$.

Solution

Using the Product Rule gives

$$\frac{dy}{dt} = e^t \cos t - e^t \sin t = e^t (\cos t - \sin t)$$

(b) Find $\dfrac{d^2y}{dt^2}$.

Solution

Using the result from the previous part we have

$$\frac{d^2y}{dt^2} = e^t (\cos t - \sin t) + e^t (-\sin t - \cos t) = -2e^t \sin t$$

(c) Show that y satisfies the **second order differential equation** $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 2y = 0$.

Solution

Using the results from parts (a) and (b) gives

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = -2e^t \sin t - 2e^t (\cos t - \sin t) + 2e^t \cos t = 0$$

Example 5

Find the derivative of each function.

(a) $f(x) = x \arctan(e^x)$

Solution

$$f'(x) = \arctan(e^x) + x\left(\frac{e^x}{1 + (e^x)^2}\right) = \arctan(e^x) + \frac{xe^x}{1 + e^{2x}}$$

(b) $g(y) = \arcsin(\sqrt{y})$

Solution

$$g'(y) = \frac{1}{\sqrt{1 - (\sqrt{y})^2}}\left(\tfrac{1}{2}y^{-1/2}\right) = \frac{1}{2\sqrt{y}\sqrt{1-y}} = \frac{1}{2\sqrt{y - y^2}}$$

(c) $w = \arctan(2 \sin v)$

Solution

$$\frac{dw}{dv} = \frac{2\cos v}{1 + (2\sin v)^2} = \frac{2\cos v}{1 + 4\sin^2 v}$$

Example 6

Find $\dfrac{dy}{dx}$ if

(a) $\dfrac{x}{x+y} = y^2$

Solution

Taking the derivative of both sides, remembering that y is a function of x and using y' in place of $\dfrac{dy}{dx}$ gives

$$\frac{(x+y) - x(1+y')}{(x+y)^2} = 2yy' \Rightarrow y - xy' = 2yy'(x+y)^2$$

Solving for y' gives

$$xy' + 2yy'(x+y)^2 = y'\left[x + 2(x+y)^2\right] = y \Rightarrow \frac{dy}{dx} = y' = \frac{y}{x+2(x+y)^2}$$

(b) $y = \tan\sqrt{x^2+y^2}$

Solution

Taking the derivative of both sides, remembering that y is a function of x and using y' in place of $\dfrac{dy}{dx}$ gives

$$\begin{aligned} y' &= \sec^2\sqrt{x^2+y^2}\left[\tfrac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x+2yy')\right] \\ &= \sec^2\sqrt{x^2+y^2}\left(\frac{x+yy'}{\sqrt{x^2+y^2}}\right) \end{aligned}$$

Solving for y' gives

$$\begin{aligned} y'\sqrt{x^2+y^2} &= x\sec^2\sqrt{x^2+y^2} + yy'\sec^2\sqrt{x^2+y^2} \\ \Rightarrow y'\left(\sqrt{x^2+y^2} - y\sec^2\sqrt{x^2+y^2}\right) &= x\sec^2\sqrt{x^2+y^2} \\ \Rightarrow \frac{dy}{dx} = y' &= \frac{x\sec^2\sqrt{x^2+y^2}}{\sqrt{x^2+y^2} - y\sec^2\sqrt{x^2+y^2}} \end{aligned}$$

(c) $xe^{xy} = y$

Solution

Taking the derivative of both sides, remembering that y is a function of x and using y' in place of $\dfrac{dy}{dx}$ gives

$$e^{xy} + xe^{xy}\left(y + xy'\right) = y' \Rightarrow e^{xy} + xye^{xy} + x^2ye^{xy} = y'$$

Solving for y' gives

$$y'\left(1 - x^2e^{xy}\right) = e^{xy}(1+xy) \Rightarrow \frac{dy}{dx} = y' = \frac{e^{xy}(1+xy)}{1-x^2e^{xy}}$$

Example 7

Consider the curve with equation $y^3 - 3xy^2 = 9 - 3x^2y$.

(a) Find an equation of the tangent line to the curve at the point $P(1,3)$.

Solution

We first need to find $\frac{dy}{dx}$ to obtain the slope of the tangent line. Using implicit differentiation gives

$$3y^2y' - 3y^2 - 6xyy' = -6xy - 3x^2y' \Rightarrow y'\left(3y^2 - 6xy - 3x^2\right) = 3y^2 - 6xy$$
$$\Rightarrow \frac{dy}{dx} = y' = \frac{y(y-2x)}{y^2 - 2xy - x^2}$$

Make sure that the point $P(1,3)$ lies on the curve. Plugging into the equation gives

$$3^3 - 3(1)\left(3^2\right) = 9 - 3\left(1^2\right)(3) \Rightarrow 27 - 27 = 9 - 9 \Rightarrow 0 = 0$$

So $P(1,3)$ lies on the curve since it satisfies the equation of the curve. The slope at $P(1,3)$ is

$$\left.\frac{dy}{dx}\right|_{x=1,y=3} = \frac{3(2-1)}{9-6+1} = \frac{3}{4}$$

So the equation of the tangent line at $P(1,3)$ is

$$y - 3 = \tfrac{3}{4}(x-1) = \tfrac{3}{4}x - \tfrac{3}{4} \Rightarrow y = \tfrac{3}{4}x + \tfrac{9}{4}$$

(b) Find the point(s) on the curve where the tangent line is horizontal.

Solution

For the tangent line to be horizontal, the slope of the tangent line must be zero. This is the case if the numerator of the derivative is zero. This gives

$$y(y-2x) = 0 \Rightarrow y = 0 \text{ or } y = 2x$$

Note that $y = 0$ is not a possible solution to the equation of the curve, so the only other possibility is $y = 2x$. Plugging this back into the equation of the curve gives

$$8x^3 - 3x\left(4x^2\right) = 9 - 3x^2(2x) \Rightarrow 2x^3 = 9 \Rightarrow x = \tfrac{1}{2}\sqrt[3]{36}$$

So the point on the curve where the tangent line is horizontal is $\left(\frac{1}{2}\sqrt[3]{36}, \sqrt[3]{36}\right)$.