Example 1

Let

$$h(x) = \sqrt{x} + x$$

(a) Use the definition of the derivative at a point

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

to find $h'(4)$.

Solution

Using the definition gives

$$h'(4) = \lim_{x \to 4} \frac{h(x) - h(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} + x - 6}{x - 4}$$

Multiplying numerator and denominator by $x - 6 - \sqrt{x}$ gives

$$\begin{aligned} h'(4) &= \lim_{x \to 4} \left(\frac{\sqrt{x} + x - 6}{x - 4}\right)\left(\frac{x - 6 - \sqrt{x}}{x - 6 - \sqrt{x}}\right) \\ &= \lim_{x \to 4} \frac{(x-6)^2 - x}{(x-4)\left(x - 6 - \sqrt{x}\right)} = \lim_{x \to 4} \frac{x^2 - 12x + 36 - x}{(x-4)\left(x - 6 - \sqrt{x}\right)} \\ &= \lim_{x \to 4} \frac{x^2 - 13x + 36}{(x-4)\left(x - 6 - \sqrt{x}\right)} = \lim_{x \to 4} \frac{(x-4)(x-9)}{(x-4)\left(x - 6 - \sqrt{x}\right)} \\ &= \lim_{x \to 4} \frac{x - 9}{x - 6 - \sqrt{x}} = \frac{4 - 9}{4 - 6 - 2} = \frac{5}{4} \end{aligned}$$

Another approach is to make the change of variables $u = \sqrt{x} \Rightarrow x = u^2$. This gives

$$\begin{aligned} h'(4) &= \lim_{u \to 2} \frac{u + u^2 - 6}{u^2 - 4} = \lim_{u \to 2} \frac{(u-2)(u+3)}{(u-2)(u+2)} \\ &= \lim_{u \to 2} \frac{u + 3}{u + 2} = \frac{5}{4} \end{aligned}$$

(b) Use your result in part (a) above to find the equation of the tangent to the curve $y = \sqrt{x} + x$ at the point on the curve where $x = 4$.

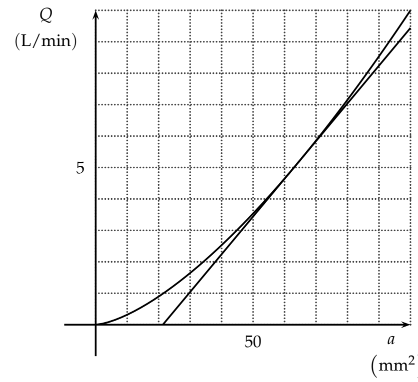
Solution

Since $h(4) = 6$, the tangent line goes through the point $(4, 6)$. So the equation is

$$y - 6 = \tfrac{5}{4}(x - 4) = \tfrac{5}{4}x - 5 \Rightarrow y = \tfrac{5}{4}x + 1$$

Example 2

The function $Q = F(a)$ gives the flow rate, in litres per minute, out of a hole in the bottom of a jug as a function of the area a of the hole in square millimetres. The graph of the function F is shown for a from 0 to $100\,\text{mm}^2$ together with the line tangent to the graph at the point where $a = 64\,\text{mm}^2$.



(a) Describe what the derivative $\frac{dQ}{da}$ represents and give its units. It is not sufficient to simply say that it is the slope of the graph at a certain point.

Solution

The derivative gives the rate of change of the flow rate with respect to the area of the hole.

(b) By estimating the slope of the tangent line estimate the value of $F'(64)$.

Solution

From the graph the tangent line goes through the points $(21, 0)$ and $(100, 9.3)$. Hence the slope, and so the value of $F'(64)$, is approximately

$$F'(64) = \left.\frac{dQ}{da}\right|_{a=64} \approx \frac{9.3 - 0}{100 - 21} = 0.117\,\frac{\text{L/min}}{\text{mm}^2}$$

(c) Explain in layman's terms what the value of the derivative in part (b) above tells you.

Solution

If the area of the hole increases from by $1\,\text{mm}^2$ from $a = 64\,\text{mm}^2$ the flow rate will increase by $0.117\,\text{L/min}$.

Example 3

Let

$$F(x) = \frac{3x}{1+2x}$$

Use the definition of the derivative

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

to find $F'(x)$.

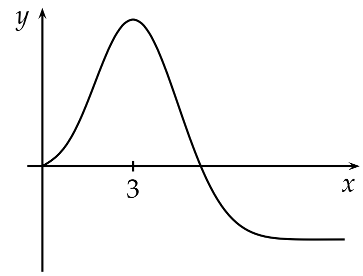
Solution

Using the definition gives

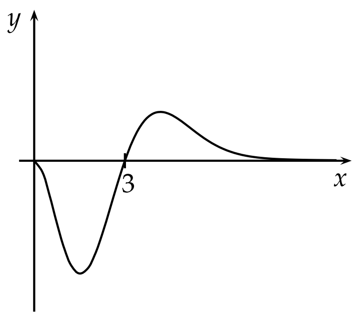
$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{\dfrac{3(x+h)}{1+2(x+h)} - \dfrac{3x}{1+2x}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{3(x+h)}{1+2(x+h)} - \frac{3x}{1+2x}\right) \\
&= \lim_{h\to 0} \frac{3}{h}\left(\frac{(x+h)(1+2x) - x(1+2x+2h)}{(1+2x+2h)(1+2x)}\right) \\
&= \lim_{h\to 0} \frac{3}{h}\left(\frac{x + 2x^2 + h + 2xh - x - 2x^2 - 2xh}{(1+2x+2h)(1+2x)}\right) \\
&= \lim_{h\to 0} \frac{3}{h}\left(\frac{h}{(1+2x+2h)(1+2x)}\right) = \lim_{h\to 0} \frac{3}{(1+2x+2h)(1+2x)} = \frac{3}{(1+2x)^2}
\end{aligned}$$

Example 4

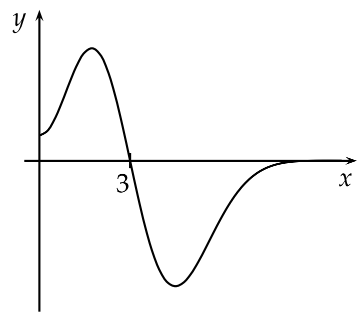
The graph of a function f is shown.



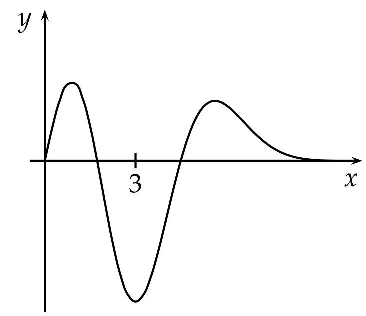
(a) The graphs of three functions are shown below. Choose the one that is the graph of the derivative of the function f.



(A)



(B)



(C)

Solution

Graph (B) is the graph of the derivative

(b) Choose the graph from the graphs in part (a) above that is the graph of the second derivative of the function f.

Solution

Graph (C) is the graph of the second derivative.

(c) Explain your choices in parts (a) and (b).

Solution

The function f is increasing, graph has positive slope, as x increases from zero and has a high point, with zero slope, at $x = 3$. Graph (B) shows positive values for x near zero and a value of zero at $x = 3$. Hence, it must be the graph of the derivative of f. Graph (B) has zero slope at $x = 0$, then positive slope until the high point in the graph to the left of $x = 3$, then negative slope through $x = 3$ until the low point on the graph to the right of $x = 3$. Hence, graph (B) is the graph of the derivative.

Graph (C) shows positive values up to a point to the left of $x = 3$, negative values through $x = 3$, going back to positive values at a point to the right of $x = 3$. Hence, graph (C) is the graph of the derivative of the function in graph (B) and so the graph of the second derivative of the function f.